

# The Detectability of Relic (Squeezed) Gravitational Waves by Laser Interferometers

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(February 7, 2008)

## Abstract

It is shown that the expected amplitudes and specific correlation properties of the relic (squeezed) gravitational wave background may allow the registration of the relic gravitational waves by the first generation of sensitive gravity-wave detectors.

PACS numbers: 98.80.Cq, 98.70.Vc, 04.30.Nk

Typeset using REVTeX

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Relic gravitational waves are inevitably generated by strong variable gravitational field of the very early Universe which parametrically (superadiabatically) amplifies the zero-point quantum oscillations of the gravitational waves [1]. The initial vacuum quantum state of each pair of waves with oppositely directed momenta evolves into a highly correlated multiparticle state known as the two-mode squeezed vacuum quantum state [2]. (For a recent review of squeezed states see [3]). In cosmological context, the squeezing manifests itself in a specific standing-wave pattern and periodic correlation functions of the generated field [4]. The main point of this paper is to show that this signature may significantly facilitate the detection of the relic (squeezed) gravitational wave background. It is possible that the appropriate data processing may allow the detection of relic gravitational waves by the forthcoming sensitive instruments, such as the initial laser interferometers in LIGO [5], VIRGO [6], GEO600 [7].

Before analyzing the theoretical predictions in light of the current experimental situation, it is necessary to explain the origin of these predictions.

We consider the cosmological gravitational wave field  $h_{ij}$  defined by the expression

$$ds^2 = a^2(\eta)[d\eta^2 - (\delta_{ij} + h_{ij})dx^i dx^j] . \quad (1)$$

The Heisenberg operator for the quantized real field  $h_{ij}$  can be written as

$$h_{ij}(\eta, \mathbf{x}) = \frac{C}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3\mathbf{n} \sum_{s=1}^2 \overset{s}{p}_{ij}(\mathbf{n}) \frac{1}{\sqrt{2n}} \left[ \overset{s}{h}_n(\eta) e^{i\mathbf{n}\mathbf{x}} \overset{s}{c}_{\mathbf{n}} + \overset{s}{h}_n^*(\eta) e^{-i\mathbf{n}\mathbf{x}} \overset{s}{c}_{\mathbf{n}}^\dagger \right], \quad (2)$$

where  $C = \sqrt{16\pi} l_{Pl}$  and the Planck length is  $l_{Pl} = (G\hbar/c^3)^{1/2}$ , the creation and annihilation operators satisfy  $\overset{s}{c}_{\mathbf{n}}|0\rangle = 0$ ,  $[\overset{s'}{c}_{\mathbf{n}}, \overset{s}{c}_{\mathbf{m}}^\dagger] = \delta_{ss'}\delta^3(\mathbf{n} - \mathbf{m})$ , and the wave number  $n$  is related to the wave vector  $\mathbf{n}$  by  $n = (\delta_{ij}n^i n^j)^{1/2}$ . The two polarisation tensors  $\overset{s}{p}_{ij}(\mathbf{n})$  ( $s = 1, 2$ ) obey the conditions

$$\overset{s}{p}_{ij}n^j = 0, \quad \overset{s}{p}_{ij}\delta^{ij} = 0, \quad \overset{s'}{p}_{ij}\overset{s}{p}^{ij} = 2\delta_{ss'}, \quad \overset{s}{p}_{ij}(-\mathbf{n}) = \overset{s}{p}_{ij}(\mathbf{n}).$$

For every wave number  $n$  and each polarisation component  $s$ , the functions  $\overset{s}{h}_n(\eta)$  have the form

$$\overset{s}{h}_n(\eta) = \frac{1}{a(\eta)} [\overset{s}{u}_n(\eta) + \overset{s}{v}_n^*(\eta)], \quad (3)$$

where  $\overset{s}{u}_n(\eta)$  and  $\overset{s}{v}_n(\eta)$  are expressed in terms of the three real functions:  $r_n$  - squeeze parameter,  $\phi_n$  - squeeze angle,  $\theta_n$  - rotation angle,

$$u_n = e^{i\theta_n} \cosh r_n, \quad v_n = e^{-i(\theta_n - 2\phi_n)} \sinh r_n. \quad (4)$$

The functions  $r_n(\eta)$ ,  $\phi_n(\eta)$ ,  $\theta_n(\eta)$  are governed by the dynamical equations [4]:

$$r'_n = \frac{a'}{a} \cos 2\phi_n, \quad \phi'_n = -n - \frac{a'}{a} \sin 2\phi_n \coth 2r_n, \quad \theta'_n = -n - \frac{a'}{a} \sin 2\phi_n \tanh r_n, \quad (5)$$

where  $' = d/d\eta$ , and the evolution begins from  $r_n = 0$ , which characterizes the initial vacuum state. The dynamical equations and their solutions are identical for both polarisation components  $s$ .

The present day values of  $r_n$  and  $\phi_n$  are essentially all we need to calculate. The mean number of particles in a two-mode squeezed state is  $2 \sinh^2 r_n$  (for each  $s$ ). This number determines the mean square amplitude of the gravitational wave field. The time behaviour of the squeeze angle  $\phi_n$  determines the time dependence of the correlation functions of the field. The amplification (that is, the growth of  $r_n$ ) governed by (5) is different for different wave numbers  $n$ . Therefore, the present day results depend on the present day frequency  $\nu$  ( $\nu = cn/2\pi a$ ) measured in  $Hz$ .

In the short-wavelength (high-frequency) regime, that is, during intervals of time when the wavelength  $\lambda(\eta) = 2\pi a/n$  is shorter than the Hubble radius  $l(\eta) = a^2/a'$ , the term  $n$  in (5) is dominant, and the functions  $\phi_n(\eta)$ ,  $\theta_n(\eta)$  are  $\phi_n = -n(\eta + \text{const})$ ,  $\theta_n = \phi_n$ . The factor  $\cos 2\phi_n$  is a quickly oscillating function of time, so the squeeze parameter  $r_n$  stays practically constant.

In the opposite, long-wavelength regime, the term  $n$  can be neglected. The function  $\phi_n$  is  $\tan \phi_n(\eta) \approx \text{const}/a^2(\eta)$ , and the squeeze angle quickly approaches one of the two values  $\phi_n = 0$  or  $\phi_n = \pi$  (an analog of the “phase bifurcation” [8]). The squeeze parameter  $r_n(\eta)$  grows with time according to

$$r_n(\eta) \approx \ln \frac{a(\eta)}{a_*} \quad , \quad (6)$$

where  $a_*$  is the value of  $a(\eta)$  when the long-wavelength regime, for a given  $n$ , begins. The final amount of  $r_n$  is

$$r_n \approx \ln \frac{a_{**}}{a_*} \quad , \quad (7)$$

where  $a_{**}$  is the value of  $a(\eta)$  when the long-wavelength regime and amplification come to the end.

After the end of amplification, the accumulated (and typically large) squeeze parameter  $r_n$  stays approximately constant. The complex functions  $\dot{u}_n^s(\eta) + \dot{v}_n^{s*}(\eta)$  become practically real, and one has  $\dot{h}_n^s(\eta) \approx \dot{h}_n^{s*}(\eta) \approx \frac{1}{a} e^{r_n} \cos \phi_n(\eta)$ . Every mode  $n$  of the field (2) takes the form of a product of a function of time and a (random, operator-valued) function of spatial coordinates, that is, the mode acquires a standing-wave pattern. The periodic dependence  $\cos \phi_n(\eta)$  is a subject of our further inquiry.

The numerical results depend on the concrete behaviour of the pump field represented by the cosmological scale factor  $a(\eta)$ . We know that the present matter-dominated stage  $a(\eta) \propto \eta^2$  was preceded by the radiation-dominated stage  $a(\eta) \propto \eta$ . The function  $a(\eta)$  describing the initial stage of expansion of the very early Universe (before the era of primordial nucleosynthesis) is not known. It is convenient to parameterize  $a(\eta)$  by power-law functions in terms of  $\eta$ , since they produce gravitational waves with power-law spectra in terms of  $\nu$  [1]. Concretely, we take  $a_i(\eta)$  at the initial stage of expansion as  $a_i(\eta) = l_o |\eta|^{1+\beta}$  where  $\eta$  grows from  $-\infty$ , and  $\beta < -1$ . From  $\eta = \eta_1$ ,  $\eta_1 < 0$ , the initial stage is followed by the radiation-dominated stage  $a_e(\eta) = l_o a_e (\eta - \eta_e)$  and then, from  $\eta = \eta_2$ , by the matter-dominated stage  $a_m(\eta) = l_o a_m (\eta - \eta_m)^2$ . The constants  $a_e$ ,  $a_m$ ,  $\eta_e$ ,  $\eta_m$  are expressed in terms of the fundamental parameters  $l_o$ ,  $\beta$  through the continuous joining of  $a(\eta)$  and  $a'(\eta)$  at  $\eta_1$ ,  $\eta_2$ . The present era is defined by the observationally known value of the Hubble radius  $l_H = c/H \approx 2 \times 10^{28}$  cm. We denote this time by  $\eta_R$  and choose  $\eta_R - \eta_m = 1$ , so that

$a(\eta_R) = 2l_H$ . The ratio  $a(\eta_R)/a(\eta_2) = z$  is believed to be around  $z = 10^4$ . Some information about  $l_o$  and  $\beta$  is provided by the data on the microwave background anisotropies [9, 10], and we will use it below.

The known function  $a(\eta)$  and equations (5) allow us to find the present day values of  $r_n$  as a function of  $n$ . Let the wave numbers  $n_H (n_H = 4\pi)$ ,  $n_m (n_m = \sqrt{z}n_H)$ ,  $n_c$  denote the waves which are leaving the long-wavelength regime at, correspondingly,  $\eta_R$ ,  $\eta_2$ ,  $\eta_1$ . The  $n_c$  is the value of  $n$  for which  $a_{**} = a_*$ . The shorter waves, with  $n > n_c$ , have never been in the amplifying long-wavelength regime. (The present day frequency  $\nu_c$  is around  $10^{10}$  Hz, as we will see below). Thus,  $r_n = 0$  for  $n_c \leq n$ ,  $r_n = \ln[(n/n_c)^\beta]$  for  $n_m \leq n \leq n_c$ ,  $r_n = \ln[(n/n_c)^{\beta-1}(n_m/n_c)]$  for  $n_H \leq n \leq n_m$ , and  $r_n = \ln[(n/n_c)^{\beta+1}(n_m/n_H)(n_c/n_H)]$  for  $n \leq n_H$ . The  $e^{r_n}$  is much larger than 1 for all frequencies  $n \ll n_c$  ( $\nu \ll \nu_c$ ).

The mean value of the field  $h_{ij}$  is zero,  $\langle 0|h_{ij}(\eta, \mathbf{x})|0 \rangle = 0$ . The variance

$$\langle 0|h_{ij}(\eta, \mathbf{x})h^{ij}(\eta, \mathbf{x})|0 \rangle \equiv \langle h^2 \rangle$$

is not zero, and it determines the mean square amplitude of the generated field - the quantity of interest for the experiment. Taking the product of two expressions (2), one can show that

$$\langle h^2 \rangle = \frac{C^2}{2\pi^2} \int_0^\infty n \sum_{s=1}^2 |h_n^s(\eta)|^2 dn = \frac{C^2}{\pi^2 a^2} \int_0^\infty n dn (\cosh 2r_n + \cos 2\phi_n \sinh 2r_n). \quad (8)$$

Eq. (8) can also be written as

$$\langle h^2 \rangle = \int_0^\infty h^2(n, \eta) \frac{dn}{n},$$

where, for the present era,

$$h(n, \eta) \approx \frac{C}{\pi} \frac{1}{a(\eta_R)} n e^{r_n} \cos \phi_n(\eta). \quad (9)$$

The further reduction of this formula gives

$$h(n) \approx A \left( \frac{n}{n_H} \right)^{\beta+2}, \quad n \leq n_H, \quad (10)$$

$$h(n, \eta) \approx A \cos \phi_n(\eta) \left( \frac{n}{n_H} \right)^\beta, \quad n_H \leq n \leq n_m, \quad (11)$$

$$h(n, \eta) \approx A \cos \phi_n(\eta) \left( \frac{n}{n_H} \right)^{\beta+1} \left( \frac{n_H}{n_m} \right), \quad n_m \leq n \leq n_c, \quad (12)$$

where

$$A = \frac{l_{Pl}}{l_o} \frac{8\sqrt{\pi} 2^{\beta+2}}{|1 + \beta|^{\beta+1}}.$$

The available information on the microwave background anisotropies [9, 10] allows us to determine the parameters  $A$  and  $\beta$ . The quadrupole anisotropy produced by the spectrum

(10) - (12) is mainly accounted for by the wave numbers near  $n_H$ . Thus, the numerical value of the quadrupole anisotropy produced by relic gravitational waves is approximately equal to  $A$ . Since (according to [11]) the quadrupole contribution of relic gravitational waves is not smaller than that produced by primordial density perturbations, this gives us  $A \approx 10^{-5}$ . We do not know experimentally whether a significant part of the quadrupole signal is indeed provided by relic gravitational waves, but we can at least assume this. The evaluation of the spectral index  $n$  of the primordial perturbations resulted in  $n = 1.2 \pm 0.3$  [10] or even in a significantly higher value  $n = 1.84 \pm 0.29$  [12] (see also [19], where one of the best fits corresponds to  $n = 1.4$ ). We interpret [13] these evaluations as an indication that the true value of  $n$  lies somewhere in the interval  $n = 1.2 \sim 1.4$  (hopefully, the planned new observational missions will determine this index more accurately). Since  $n \equiv 2\beta + 5$ , this gives us the parameter  $\beta$  in the interval  $\beta = -1.9 \sim -1.8$ . In fact, the value  $n = 1.4$  ( $\beta = -1.8$ ) is the largest one for which the entire theoretical approach is well posed, since this value requires the pumping field to be too strong: the Hubble radius at the end of the initial stage would have been only a little larger than  $l_{Pl}$ . With the adopted  $A$  and  $\beta$ , the frequency  $\nu_c$  falls in the region  $\nu_c = 10^{10} Hz$  or higher. An allowed intermediate stage of expansion governed by a “stiff” matter [14] can affect the spectral slope at frequencies somewhat lower than  $\nu_c$ , but still outside the interval accessible to the existing detection techniques. The details of a short transition from the initial stage to the radiation-dominated stage are irrelevant for our discussion, since they can alter the signal only at frequencies around  $\nu_c$ . It is worth recalling [13] that a confirmation of any  $n > 1$  ( $\beta > -2$ ) would mean that the very early Universe was not driven by a scalar field - the cornerstone of inflationary considerations - because the  $n > 1$  ( $\beta > -2$ ) requires the effective equation of state at the initial stage to be  $\epsilon + p < 0$ , but this cannot be accommodated by any scalar field with whichever scalar field potential. Obviously, the available data do not prove yet that  $n > 1$  but this possibility is not ruled out either. It is important to emphasize that the existing disagreement (see [11]) with regard to the validity of the inflationary prediction of infinitely large amplitudes of density perturbations for the part of the spectrum with the spectral index  $n = 1$  ( $\beta = -2$ ) does not affect the assumptions and conclusions of this paper which analyses the cases  $\beta > -2$ .

We switch now from cosmology to experimental predictions in terms of laboratory frequencies  $\nu$  and intervals of time  $t$  ( $c dt = a(\eta_R) d\eta$ ). Formula (12) translates into

$$h(\nu, t) \approx 10^{-7} \cos[2\pi\nu(t - t_\nu)] \left( \frac{\nu}{\nu_H} \right)^{\beta+1}, \quad (13)$$

where  $\nu_H = 10^{-18} Hz$ , and  $t_\nu$  is a deterministic (not random) function of frequency which does not vary significantly on the intervals  $\Delta\nu \approx \nu$ . We take  $\nu = 10^2 Hz$  as the representative frequency for the ground-based laser interferometers. The expected sensitivity of the initial instruments at  $\nu = 10^2 Hz$  is  $h_{ex} = 10^{-21}$  or better. The theoretical prediction at this frequency, following from (13), is  $h_{th} = 10^{-23}$  for  $\beta = -1.8$ , and  $h_{th} = 10^{-25}$  for  $\beta = -1.9$ . Therefore, the gap between the signal and noise levels is from 2 to 4 orders of magnitude. This gap should be covered by a sufficiently long observation time  $\tau$ . The duration  $\tau$  depends on whether the signal has any temporal signature known in advance, or not.

It appears that the periodic structure (13) should survive in the instrumental window of sensitivity from  $\nu_1$  (minimal frequency) to  $\nu_2$  (maximal frequency). The mean square value of the field in this window is

$$\int_{\nu_1}^{\nu_2} h^2(\nu, t) \frac{d\nu}{\nu} = 10^{-14} \frac{1}{\nu_H^{2\beta+2}} \int_{\nu_1}^{\nu_2} \cos^2[2\pi\nu(t - t_\nu)] \nu^{2\beta+1} d\nu. \quad (14)$$

Because of the strong dependence of the integrand on frequency,  $\nu^{-2.6}$  or  $\nu^{-2.8}$ , the integral (14) is determined by its lower limit. This gives

$$\int_{\nu_1}^{\nu_2} h^2(\nu, t) \frac{d\nu}{\nu} \approx 10^{-14} \left( \frac{\nu_1}{\nu_H} \right)^{2\beta+2} \cos^2[2\pi\nu_1(t - t_1)]. \quad (15)$$

The explicit time dependence of the variance of the field, or, in other words, the explicit time dependence of the (zero-lag) temporal correlation function of the field, demonstrates that we are dealing with a non-stationary process (a consequence of squeezing and severe reduction of the phase uncertainty). Apparently, the search through the data should be based on the periodic structure at  $\nu = \nu_1$ .

The response of an instrument to the incoming radiation is  $s(t) = F_{ij} h^{ij}$  where  $F_{ij}$  depends on the position and orientation of the instrument. The cross correlation of responses from two instruments  $\langle 0 | s_1(t) s_2(t) | 0 \rangle$  will involve the overlap reduction function [15 - 18], which we assume to be not much smaller than 1 [17]. The essential part of the cross correlation will be determined by an expression of the same form as (15).

The signal to noise ratio  $S/N$  in the measurement of the amplitude of a signal with no specific known features increases as  $(\tau\nu_1)^{1/4}$ . If the signal has known features exploited by the matched filtering technique, the  $S/N$  increases as  $(\tau\nu_1)^{1/2}$ . The guaranteed law  $(\tau\nu_1)^{1/4}$  requires a reasonably short time  $\tau = 10^6$  sec to improve the  $S/N$  by two orders of magnitude and to reach the level of the predicted signal with the extreme spectral index  $\beta = -1.8$ . If the law  $(\tau\nu_1)^{1/2}$  can be implemented, the same observation time  $\tau = 10^6$  sec will allow the registration of the signal with the conservative spectral index  $\beta = -1.9$ . Even an intermediate law between  $(\tau\nu_1)^{1/4}$  and  $(\tau\nu_1)^{1/2}$  may turn out to be sufficient. For the network of ground-based interferometers the expected  $\nu_1$  is around  $30Hz$ , but we have used  $\nu_1 = 10^2 Hz$  for a conservative estimate of  $\tau$ . If the matched filtering technique can indeed be used, it can prove sufficient to have data from a single interferometer.

For the frequency intervals covered by space intereferometers, solid-state detectors, and electromagnetic detectors, the expected results follow from the same formula (13) and have been briefly discussed elsewhere [13].

In conclusion, the detection of relic (squeezed) gravitational waves may be awaiting only the first generation of sensitive instruments and an appropriate data processing strategy.

I appreciate useful discussions with S. Dhurandhar and B. Sathyaprakash.

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[1 ] L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **67**, 825 (1974) [JETP **40**, 409 (1975)]; Ann. NY Acad. Sci. **302**, 439 (1977).

[2 ] L. P. Grishchuk and Yu. V. Sidorov, Class. Quant. Grav. **6**, L161 (1989); Phys. Rev. **D42**, 3413 (1990).

[3 ] P. L. Knight, in *Quantum Fluctuations*, Eds. S. Reynaud, E. Giacobino, and J. Zinn-Justin, (Elsevier Science) 1997, p. 5.

- [4 ] L. P. Grishchuk, in *Workshop on Squeezed States and Uncertainty Relations*, NASA Conf. Publ. **3135**, 1992, p. 329; Class. Quant. Grav. **10**, 2449 (1993).
- [5 ] A. Abramovici *et. al.*, Science **256**, 325 (1992).
- [6 ] C. Bradaschia *et. al.*, Nucl. Instrum. and Methods **A289**, 518 (1990).
- [7 ] J. Hough and K. Danzmann *et. al.*, GEO600 Proposal, 1994.
- [8 ] W. Schleich and J. A. Wheeler, J. Opt. Soc. Am **B4**, 1715 (1987); W. Schleich *et. al.*, Phys. Rev. **A40**, 7405 (1989).
- [9 ] G. F. Smoot *et. al.*, Astroph. J. **396**, L1 (1992).
- [10 ] C. L. Bennet *et. al.*, Astroph. J. **464**, L1 (1996).
- [11 ] L. P. Grishchuk, Phys. Rev. **D50**, 7154 (1994); in *Current Topics in Astrofundamental Physics: Primordial Cosmology*, Eds. N. Sanchez and A. Zichichi, (Kluwer Academic) 1998, p. 539; Report gr-qc/9801011.
- [12 ] A. A. Brukhanov *et. al.*, Report astro-ph/9512151.
- [13 ] L. P. Grishchuk, Class. Quant. Grav. **14**, 1445 (1997).
- [14 ] M. Giovannini, Phys. Rev. **D58**, 1 September 1998 (to appear).
- [15 ] P. F. Michelson, Mon. Not. R. astr. Soc. **227**, 933 (1987).
- [16 ] N. L. Christensen, Phys. Rev. **D46**, 5250 (1992).
- [17 ] E. E. Flanagan, Phys. Rev. **D48**, 2389 (1993).
- [18 ] B. Allen, in *Relativistic Gravitation and Gravitational Radiation*, Eds. J-A. Marck and J-P. Lasota (CUP, 1997) p. 373.
- [19 ] M. Tegmark, Report astro-ph/9809201